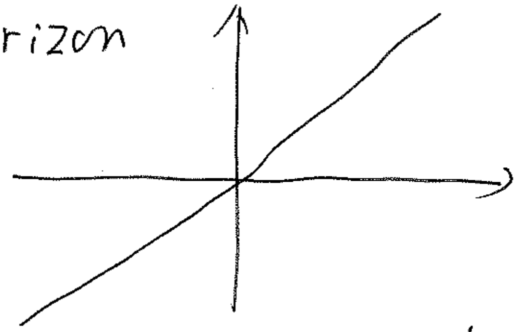
Horizon



Question: how to describe $\mathbb{R}^2 \sqcup$ Horizon ?

$\mathbb{RP}^1$ = the set of lines in $\mathbb{R}^2$ passing $\vec{0}$

$= \{(x, y) \in \mathbb{R}^2 - \vec{0}\}/\sim$, $(x, y) \sim (x', y')$ iff

$\exists\ \lambda \neq 0,\ \lambda \in \mathbb{R}^1$ such that $(x, y) = \lambda(x', y')$

$[x, y] \triangleq$ the equivalence class of $(x, y) \in \mathbb{R}^2 - \{\vec{0}\}$

So $\mathbb{RP}^1 = \{[x, y] \mid (x, y) \neq \vec{0},\ [x, y] = [\lambda x, \lambda y]\}$

$\mathbb{RP}^1 \cong S^1$.

Generalization: $\mathbb{RP}^n$ = the set of lines in $\mathbb{R}^{n+1}$ passing $\vec{0}$

$= \{[x_0, x_1, \cdots x_n] \mid [x_0, \dots x_n] = [\lambda x_0, \dots, \lambda x_n],\ (x_0, \cdots x_n) \neq \vec{0}\}$

$= \{(x_0, x_1, \dots x_n) \in \mathbb{R}^{n+1} - \vec{0}\}/\sim$: $(x_0, \dots x_n) \sim (x_0', \dots, x_n')$

iff $(x_0, \cdots x_n) = \lambda(x_0', \dots, x_n')$ for some $\lambda \in \mathbb{R}^1 - \{0\}$

Look at $\mathbb{RP}^2$: $U_0 \triangleq \{[x_0, x_1, x_2] \in \mathbb{RP}^2 \mid x_0 \neq 0\}$

$= \{[1, \frac{x_1}{x_0}, \frac{x_2}{x_0}]\} \xrightarrow{\varphi_0} (\frac{x_1}{x_0}, \frac{x_2}{x_0}) \in \mathbb{R}^2$. We also have the inverse map $\mathbb{R}^2 \xrightarrow{\psi_0} U_0 \subset \mathbb{RP}^2$, $(x, y) \longrightarrow [1, x, y]$

The complement of U_0 in $\mathbb{RP}^2$ is $H_0 = \{[0, x_1, x_2] \in \mathbb{RP}^2\} \cong \mathbb{RP}^1$

$(x, y) \longrightarrow [1, x, y]$ as $\lambda \to \infty$

$(\lambda a_1, \lambda a_2) \longrightarrow [1, \lambda a_1, \lambda a_2] = [\frac{1}{\lambda}, a_1, a_2] \longrightarrow [0, a_1, a_2]$

So $[0, a_1, a_2]$ is the "vanishing point" of lines $\parallel (a_1, a_2)$ in $\mathbb{R}^2$.

We can also consider $U_1 = \{[x_0, x_1, x_2] \mid x_1 \neq 0\} \subset \mathbb{RP}^2$ and $U_2 = \{[x_0, x_1, x_2] \mid x_2 \neq 0\} \subset \mathbb{RP}^2$.

In general, $\mathbb{RP}^n \supset U_i = \{[x_0, x_1, \dots, x_n] \in \mathbb{RP}^n \mid x_i \neq 0\} \cong \mathbb{R}^n$,

$\mathbb{RP}^n - U_i \cong \mathbb{RP}^{n-1}$. $(x_0, \dots, x_n) \xrightarrow{\pi} [x_0, \dots, x_n]$

$\mathbb{R}^{n+1} \supset S^n = \{ x_0^2 + \dots + x_n^2 = 1 \} \xrightarrow{\pi} \mathbb{RP}^n$

For $(x_0, \dots, x_n)$ with $x_0^2 + \dots + x_n^2 = 1$, $\pi^{-1}([x_0, \dots, x_n]) = \{(\lambda x_0, \dots, \lambda x_n) \mid |\lambda| = 1\}$. Thus S^n is a double cover of $\mathbb{RP}^n$. Thus $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$ for $n \geq 2$.

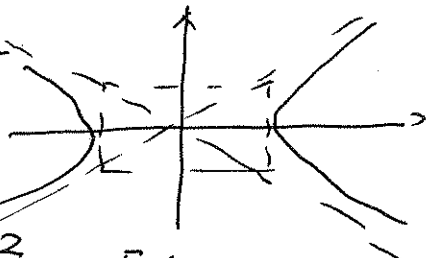
Therefore $\mathbb{RP}^n$ is compact

Algebraic aspect:

In $\mathbb{R}^2$, we have (1) lines (2) conic sections:

ellipse, hyperbola)(, and parabola U.

Consider the hyperbola



$C: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$\mathbb{R}^2 \hookrightarrow U_0 \subset \mathbb{RP}^2$ $[x_0, x_1, x_2] \in U_0 \Rightarrow [x_0, x_1, x_2] = [1, \frac{x_1}{x_0}, \frac{x_2}{x_0}]$

$(x, y) \to [1, x, y]$ So C in U_0 can be written as

$$\frac{\left(\frac{x_1}{x_0}\right)^2}{a^2} - \frac{\left(\frac{x_2}{x_0}\right)^2}{b^2} = 1 \quad \text{or equivalently} \quad \begin{cases} \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - x_0^2 = 0 \\ x_0 \neq 0 \end{cases}$$

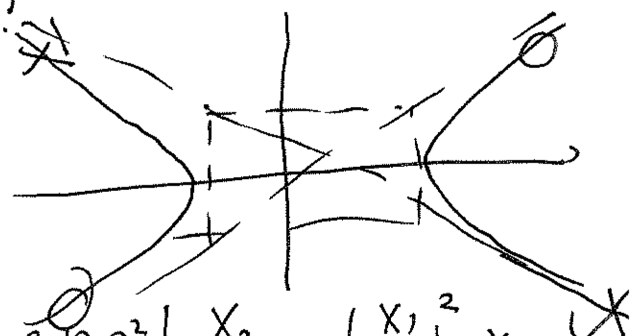
The polynomial $\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - x_0^2$ cannot define a function on $\mathbb{RP}^2$, but the zero locus $\{\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - x_0^2 = 0\}$ in $\mathbb{RP}^2$ is well-defined.

$$\bar{C} \triangleq \{[x_0, x_1, x_2] \in \mathbb{RP}^2 \mid \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - x_0^2 = 0\}.$$

Then $C \subset \bar{C}$. $\bar{C} - C = \begin{cases} \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - x_0^2 = 0 \\ x_0 = 0 \end{cases}$

$$= \begin{cases} \dfrac{x_1^2}{a^2} - \dfrac{x_2^2}{b^2} = 0 \\ x_0 = 0 \end{cases} = \begin{cases} x_1 = \pm\dfrac{a}{b} x_2 \\ x_0 = 0 \end{cases} \Rightarrow$$ $\overline{C} - C$ consists of two points $[0, a, b]$ & $[0, a, -b]$.

$[0.\ a,\ b]$ is the "infinity point" corresponding to the line $bx - ay = 0$ in $\mathbb{R}^2$, which is one of the asymptote of the hyperbola, $[0.\ a, -b]$ corresponds to the other asymptote of the hyperbola.



So $\overline{C} \cong S^1$

How about the parabola?

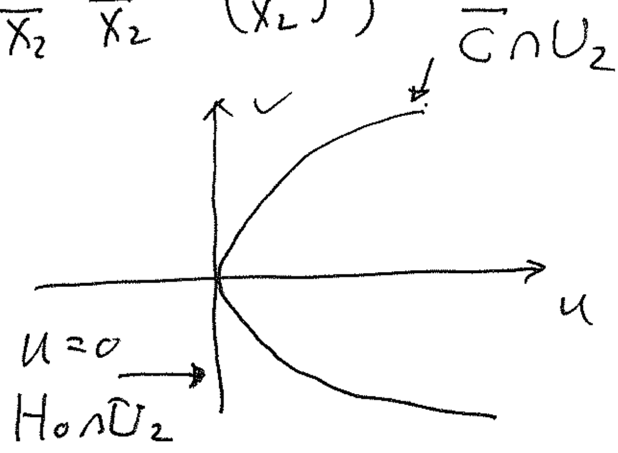
$C: y = x^2 \longrightarrow \{[x_0.\ x_1\ x_2] \in \mathbb{RP}^2 \mid \frac{x_2}{x_0} = \left(\frac{x_1}{x_0}\right)^2,\ x_0 \neq 0\}$.

$\overline{C} \cong \{[x_0, x_1 . x_2] \in \mathbb{RP}^2 \mid x_2 x_0 = x_1^2\}$.

$$\overline{C} \cap U_2 = \left\{ \Big[\underset{\substack{\shortparallel \\ u}}{\frac{x_0}{x_2}},\ \underset{\substack{\shortparallel \\ v}}{\frac{x_1}{x_2}},\ \underset{\substack{\shortparallel \\ 1}}{\frac{x_2}{x_2}} \Big] \;\middle|\; \frac{x_2}{x_2}\frac{x_0}{x_2} = \left(\frac{x_1}{x_2}\right)^2 \right\}$$

$= \{(u.v) \mid u = v^2\}$.

$H_0 \cap U_2 = \{\frac{x_0}{x_2} = 0\} = \{u = 0\}$



Thus $\overline{C} - C = \{[0, 1, 0]\}$

$\overline{C} \cong S^1$.

Homogenization and de-homogenization.

$f(y_1, \dots, y_n)$: a polynomial of degree d.

$\longrightarrow F \triangleq x_0^d f(\frac{x_1}{x_0}, \dots \frac{x_n}{x_0})$ is a homogeneous polynomial of degree d.

$F(x_0, x_1, \dots x_n)$ is a homogeneous polynomial of degree d

iff $F(\lambda x_0, \lambda x_1, \dots, \lambda x_n) = \lambda^d F(x_0, x_1, \dots, x_n)$

$X = \{f = 0\} \subset \mathbb{R}^n$, $\overline{X} = \{F = 0\} \subset \mathbb{RP}^n$. Then $\overline{X} \cap U_0 = X$

$\mathbb{R}^n \ni (y_1, \dots y_n) \longrightarrow [1, y_1, \dots y_n] \in U_0 \subset \mathbb{RP}^n$.

For a homogeneous polynomial $F(x_0, \dots x_n)$, we can de-homogenize F to get a polynomial $f(y_1, \dots y_n) = F(1, y_1, \dots y_n)$ of degree d.

Complex projective space

$\mathbb{CP}^n$ = the set of lines in $\mathbb{C}^{n+1}$ passing $\vec{0}$

$= \left\{ [z_0, \dots, z_n] \;\middle|\; \begin{array}{l} (z_0, \dots, z_n) \neq \vec{0} \\ [z_0, \dots, z_n] = [\lambda z_0, \dots, \lambda z_n] \end{array} \right\}$.

$\mathbb{C}^n \xrightarrow[\cong]{\varphi_0} U_0 = \{[z_0, z_1, \dots z_n] \mid z_0 \neq 0\} \subset \mathbb{CP}^n$.

$\mathbb{CP}^n - U_0 \cong \mathbb{CP}^{n-1}$ $\qquad$ $\mathbb{CP}^1 = S^2$, two dim. sphere.

Blow up. $\mathbb{A}^n_k = \begin{cases} \mathbb{R}^n & \text{if } k = \mathbb{R} \\ \mathbb{C}^n & \text{if } k = \mathbb{C}. \end{cases}$

$\mathbb{P}^n_k = \begin{cases} \mathbb{RP}^n & \text{if } k = \mathbb{R} \\ \mathbb{CP}^n & \text{if } k = \mathbb{C} \end{cases}$

Consider a subset X: $\mathbb{A}^n_k \times \mathbb{P}^{n-1}_k \supset X$

$$X = \{(a_1, \cdots, a_n) \times [l_1, \cdots, l_n] \mid (a_1, \cdots a_n) = \lambda (l_1, \cdots l_n),\ \lambda \in k\}$$

$\pi \downarrow$

$(a_1, \cdots, a_n)$.

$$\pi^{-1}((a_1, \cdots a_n)) = \begin{cases} \vec{0} \times \mathbb{P}^{n-1}_k & \text{if } (a_1, \ldots a_n) = \vec{0} \\ \underline{(a_1, \ldots a_n) \times [l_1, \ldots l_n]} & \text{if } (a_1, \ldots a_n) \neq \vec{0} \end{cases}$$

$\Uparrow$

Since $(a_1, \ldots a_n) \neq \vec{0}$, $\lambda \neq 0$.

So $(l_1, \ldots l_n) = (\frac{1}{\lambda} a_1, \ldots \frac{1}{\lambda} a_n)$,

Thus $[l_1, \ldots l_n] = [\frac{1}{\lambda} a_1, \cdots, \frac{1}{\lambda} a_n]$

$= [a_1, \ldots a_n]$.

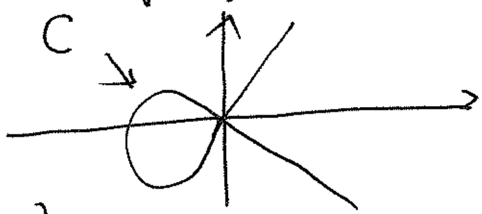
Therefore, $\pi^{-1}((a_1 \ldots a_n))$ = a point

$= (a_1 \ldots a_n) \times [a_1, \ldots a_n]$ if $(a_1 \ldots a_n) \neq \vec{0}$.

So away from $\vec{0}$,

$X - \pi^{-1}(\vec{0}) \cong \mathbb{A}^n_k - \{\vec{0}\}$

$\pi^{-1}(\vec{0}) = \mathbb{P}^{n-1}_k$. X is called the blowup of $\mathbb{A}^n_k$ at $\vec{0}$.



Ex: $C: x^2 - y^2 + x^3 = 0$ in $\mathbb{A}^2_k$

$$\pi^{-1}(C) = \{(x, y) \times [l_1, l_2] \mid x^2 - y^2 + x^3 = 0,\ (x, y) = (\lambda l_1, \lambda l_2)\}$$

Let $X_1 = \{(x, y) \times [l_1, l_2] \in X \mid l_1 \neq 0\}$

$$\pi^{-1}(C) \cap X_1 = \{(x, y) \times [l_1, l_2] \mid x^2 - y^2 + x^3 = 0,\ \lambda = \frac{x}{l_1},\ y = \lambda l_2 \Leftrightarrow y = x \frac{l_2}{l_1}\}$$

Let $u = \frac{l_2}{l_1}$ So $X_1 \cong \mathbb{R}^2$ with coordinates (x, u).

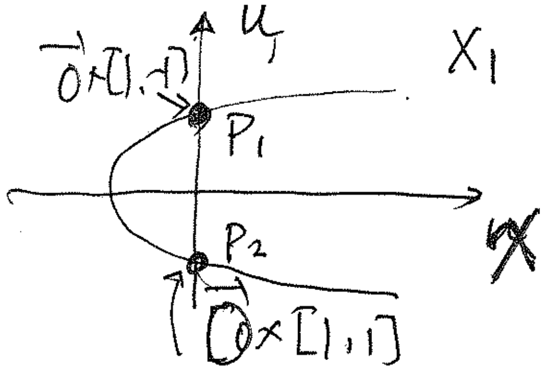
$\pi^{-1}(C) \cap X_1 = \{x^2 - (xu)^2 + x^3 = 0\} = \{x^2(1 - u^2 + x) = 0\}$

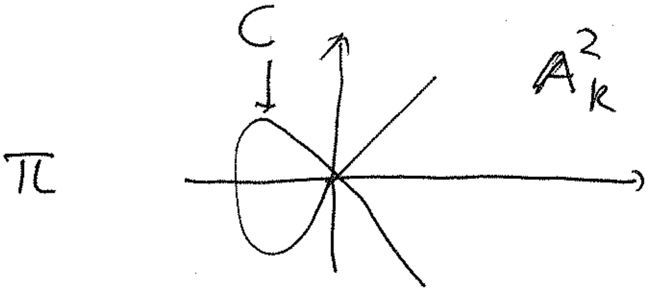
$= \{x^2 = 0\} \cup \{1 - u^2 + x = 0\} = (\pi^{-1}(\vec{0}) \cap X_1) \cup (\tilde{C} \cap X_1)$

where $\tilde{C} \cap X_1 = \{1 - u^2 + x = 0\} \leftarrow$ smooth curve.

Intersection of $\tilde{C}$ with $\pi^{-1}(\vec{0})$ is: $\begin{cases} 1 - u^2 + x = 0 \\ x = 0 \end{cases}$

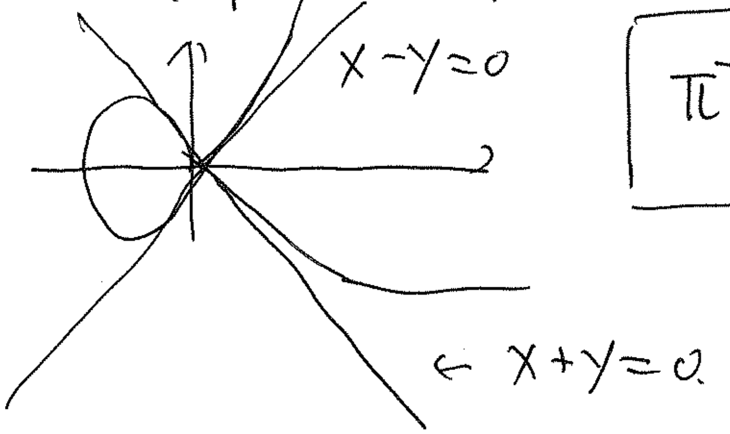
$\Rightarrow \begin{cases} u = \pm 1 \\ x = 0 \end{cases} \Rightarrow \vec{0} \times [1, 1]$ and $\vec{0} \times [1, -1]$





$\pi(P_1) = \pi(P_2) = \vec{0}$

So the intersection of $\tilde{C}$ with $\pi^{-1}(\vec{0})$ has two points. P_1 represents the tangent line $x - y = 0$ and P_2 corresponds to the tangent line $x + y = 0$



$$\boxed{\pi^{-1}(C) = \tilde{C} \cup \pi^{-1}(\vec{0})}$$

Homework: Use the blowup to find the intersection of $\tilde{C}$ with $\pi^{-1}(\vec{0})$

(1) $C: \ xy - x^6 - y^6 = 0$

(2) $C: \ x^3 = y^2 + x^4 + y^4$